The new generation of interest-rate derivatives models: The Libor and swap market models

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These slides are based on Chapters 6,7 and 8 of Brigo and Mercurio’s book, “Interest-Rate Models: Theory and Practice”, Springer Verlag, 2001.

The reader is referred to such book for a rigorous treatment and references.
Structure of the Talk

- Introduction to the interest-rate markets (CAPS and SWAPTIONS)
- Giving rigor to Black’s formulas: The LFM and LSM market models in general
- Theoretical incompatibility of LSM and LFM
- Practical compatibility of LSM and LFM?
- Choosing a LFM model: parameterizing instantaneous covariances
- Joint calibration of the LFM to Caps and Swaptions
- Evolution of the term structure of volatility and terminal correlations
- Practical Examples of Calibration and Diagnostics
- Hints at smile modeling and more advanced issues
Introduction to the interest-rate markets (CAPS and SWAPTIONS)

**Bank-account:** $dB(t) = r_t B(t) \, dt, \quad B(0) = 1$

$r_t \geq 0$ is the instantaneous accruing rate

Value at $t$ of one unit of currency available at $T$ is

Discount Factor $D(t, T) = \frac{B(t)}{B(T)} = \exp \left( - \int_t^T r_s \, ds \right)$.

Some particular forms (stochastic differential equations) of possible evolutions for $r$ constitute the **short-rate models** (Vasicek, CIR, BDT, BK, HW...)

A $T$–maturity zero–coupon bond is a contract which guarantees the payment of one unit of currency at time $T$. The contract value at time $t < T$ is denoted by $P(t, T)$:

$$P(T, T) = 1 \quad P(t, T) = \tilde{E}_t D(t, T)$$

All kind of rates can be expressed in terms of zero–coupon bonds and vice-versa. ZCB’s can be used as fundamental quantities.
Introduction to the interest-rate markets (CAPS and SWAPTIONS)

The spot–Libor rate at time $t$ for the maturity $T$ is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t, T)$ units of currency at time $t$, when accruing occurs proportionally to the investment time.

\[ P(t, T)(1 + \tau(t, T) L(t, T)) = 1, \quad L(t, T) = \frac{1 - P(t, T)}{\tau(t, T) P(t, T)}. \]

Notice:

\[ r(t) = \lim_{T \to t^+} L(t, T). \]

The zero–coupon curve (often referred to as “yield curve”) at time $t$ is the graph of the function

\[ T \mapsto L(t, T). \]

This function is also called the term structure of interest rates at time $t$. This is a snapshot at time $t$. Time is frozen at $t$. This does not involve a dynamical model.
Introduction to the interest-rate markets (CAPS and SWAPTIONS)

A forward rate agreement FRA is a contract involving three time instants: The current time $t$, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest rate payment for the period $T \mapsto S$ with fixed rate $K$ at maturity $S$ against an interest rate payment over the same period with rate $L(T, S)$.

Basically, this contract allows one to lock-in the interest rate between $T$ and $S$ at a desired value $K$.

By easy static no-arbitrage arguments:

$$FRA(t, T, S, K) = P(t, S)\tau(T, S)K - P(t, T) + P(t, S).$$

The value of $K$ which makes the contract fair ($=0$) is the forward LIBOR interest rate prevailing at time $t$ for the expiry $T$ and maturity $S$: $K = F(t; T, S)$.

$$F(t; T, S) := \frac{1}{\tau(T, S)} \left( \frac{P(t, T)}{P(t, S)} - 1 \right) = E_t^S L(T, S).$$
Introduction to the interest-rate markets (CAPS and SWAPTIONS)

A Payer Interest Rate Swap (PFS) is a contract that exchanges payments between two differently indexed legs, starting from a future time–instant. At future dates $T_{\alpha+1}, ..., T_\beta$,

$$\rightarrow \tau_j K \rightarrow$$

at $T_j$:

Fixed Leg

$$\leftarrow \tau_j L(T_{j-1}, T_j) \leftarrow$$

$$\tau_j F(T_\alpha; T_{j-1}, T_j)$$

The discounted payoff at a time $t < T_\alpha$ of an IRS is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) \tau_i(K - L(T_{i-1}, T_i)),$$

or alternatively

$$D(t, T_\alpha) \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) \tau_i(K - F(T_\alpha; T_{i-1}, T_i)).$$

IRS can be valued as a collection of FRAs. The value $K = S_{\alpha, \beta}(t)$ which makes IRS a fair (=0) contract is the forward swap rate.
**forward swap rate:** This is that value $S_{\alpha, \beta}(t)$ of $K$ for which $IRS(t, [T_\alpha, \ldots, T_\beta], K) = 0$. $S_{\alpha, \beta}(t) =$

$$S_{\alpha, \beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)} = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}.$$

A **cap** can be seen as a payer IRS where each exchange payment is executed only if it has positive value.

Cap discounted payoff: $\sum_{i=\alpha+1}^{\beta} D(t, T_i) \tau_i (L(T_{i-1}, T_i) - K)^+.$

Suppose a company is Libor–indebted and has to pay at $T_{\alpha+1}, \ldots, T_\beta$ the Libor rates resetting at $T_\alpha, \ldots, T_{\beta-1}$. The company has a view that libor rates will increase in the future, and wishes to protect itself

buy a cap: $(L - K)^+ \rightarrow^{CAP} \text{Company} \rightarrow^{DEBT} L$

or  $\text{Company} \rightarrow^{NET} L - (L - K)^+ = \min(L, K)$

The company pays at most $K$ at each payment date.
A cap contract can be decomposed additively:

Indeed, the discounted payoff is a sum of terms (caplets)

$$D(t, T_i) \tau_i (L(T_{i-1}, T_i) - K)^+$$.

Each caplet can be evaluated separately, and the corresponding values can be added to obtain the cap price (notice the “call option” structure!).

Finally, we introduce options on IRS’s (swaptions).

A (payer) swaption is a contract giving the right to enter at a future time a (payer) IRS.

The time of possible entrance is the maturity.

Usually maturity is first reset of underlying IRS.

IRS value at its first reset date $T_\alpha$, i.e. at maturity,

$$\tilde{E}_{T_\alpha} \sum_{i=\alpha+1}^{\beta} D(T_\alpha, T_i) \tau_i (L(T_{i-1}, T_i) - K) =$$

$$\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) \tau_i (F(T_{\alpha}; T_{i-1}, T_i) - K) =$$

$$= C_{\alpha,\beta}(T_\alpha)(S_{\alpha,\beta}(T_\alpha) - K)$$.
The option will be exercised only if this IRS value is positive. There results the payer–swaption discounted–payoff at time $t$:

$$D(t, T_\alpha)C_{\alpha,\beta}(T_\alpha)(S_{\alpha,\beta}(T_\alpha) - K)^+ =$$

$$D(t, T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) \tau_i(F(T_\alpha; T_{i-1}, T_i) - K)\right)^+$$

Unlike Caps, this payoff cannot be decomposed additively.

Caps can be decomposed in caplets, each with a single fwd rate. Caps: Deal with each caplet separately, and put results together.

Only marginal distributions of different fwd rates are involved.

Not so with swaptions: The summation is inside the positive part operator $(\cdot)^+$, and not outside.

With swaptions we will need to consider the joint action of the rates involved in the contract.

The correlation between rates is fundamental in handling swaptions, contrary to the cap case.
Giving rigor to Black’s formulas: The LFM market model in general

Recall measure $Q^U$ associated with numeraire $U$ (Risk–neutral measure $\tilde{Q} = Q^B$).

$A/U$, with $A$ a tradable asset, is a $Q^U$-martingale.

Caps: Rigorous derivation of Black’s formula.

Take $U = P(\cdot, T_i)$, $Q^U = Q^i$. Since

$$F(t; T_{i-1}, T_i) = \left(\frac{1}{\tau_i}\right)\left(\frac{P(t, T_{i-1}) - P(t, T_i)}{P(t, T_i)}\right),$$

$F(t; T_{i-1}, T_i) =: F_i(t)$ is a $Q^i$-martingale. Take

$$dF_i(t) = \sigma_i(t)F_i(t)dZ_i(t), \quad Q^i, \quad t \leq T_{i-1}.$$

This is the Lognormal Forward–Libor Model (LFM).

Consider the discounted $T_{k-1} – caplet$

$$\left(F_k(T_{k-1}) - K\right)^+ B(0)/B(T_k)$$

The LIBOR and SWAP market model
**LFM:** \( dF_k(t) = \sigma_k(t)F_k(t)dZ_k(t), \quad Q^k, \quad t \leq T_{k-1}. \)

The price at the time 0 of the single caplet is

\[
B(0)\tilde{E}\left[\frac{(F_k(T_{k-1}) - K)^+}{B(T_k)}\right] = \\
= P(0, T_k) E^k\left[\frac{(F_k(T_{k-1}) - K)^+}{P(T_k, T_k)}\right] = \ldots \\
= P(0, T_k) \text{B\&S}(F_k(0), K, v_k)
\]

\[
v_{T_{k-1}\text{-caplet}}^2 = \frac{v_k^2}{T_{k-1}} = \frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma_k(t)^2 dt
\]

Dynamics of \( F_k = F(\cdot, T_{k-1}, T_k) \) under \( Q^i \neq Q^k \) in the cases \( i < k(t \leq T_i) \) and \( i > k(t \leq T_{k-1}) \) are, respectively,

\[
dF_k(t) = \sigma_k(t)F_k \sum_{j=i+1}^{k} \frac{\rho_{k,j} \tau_j \sigma_j F_j}{1 + \tau_j F_j} dt + \sigma_k(t)F_k(t)dZ_k(t),
\]

\[
dF_k(t) = -\sigma_k(t)F_k \sum_{j=k+1}^{i} \frac{\rho_{k,j} \tau_j \sigma_j F_j}{1 + \tau_j F_j} dt + \sigma_k(t)F_k(t)dZ_k(t).
\]

where \( dZ_k dZ_j = \rho_{k,j} dt \). Unknown distributions.

**Notation:** \( dF_k = \mu^i_k F_k dt + \sigma_k F_k dZ^i_k \).
Similarly, Black's formula for swaptions becomes rigorous by taking as numeraire

\[
U = C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i), \quad Q^U = Q^{\alpha,\beta}
\]

\[
S_{\alpha,\beta}(t) = \frac{P(t, T_{\alpha}) - P(t, T_{\beta})}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}
\]

so that \( S_{\alpha,\beta} \) is a martingale under \( Q^{\alpha,\beta} \). Take

\[
d S_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t) \, dW_t^{\alpha,\beta}, \quad Q^{\alpha,\beta} \text{ (LSM)}, \text{ so that}
\]

\[
\tilde{E} \left( (S_{\alpha,\beta}(T_{\alpha}) - K)^+ \, C_{\alpha,\beta}(T_{\alpha}) B(0)/B(T_{\alpha}) \right) = \]

\[
= C_{\alpha,\beta}(0) \, E^{\alpha,\beta} (S_{\alpha,\beta}(T_{\alpha}) - K)^+ \]

\[
= C_{\alpha,\beta}(0) \, B&S (S_{\alpha,\beta}(0), K, v_{\alpha,\beta}(T_{\alpha})) ,
\]

\[
v^{2}_{\alpha,\beta}(T) = \int_{0}^{T} (\sigma^{(\alpha,\beta)}(t))^2 \, dt .
\]
Theoretical incompatibility LSM / LFM

Recall LFM: \( dF_i(t) = \sigma_i(t) F_i(t) dZ_i(t), \ Q^i \),

LSM: \( dS_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t) \ dW_t, \ Q^{\alpha,\beta} \). (1)

Precisely: Can each \( F_i \) be lognormal under \( Q^i \) and \( S_{\alpha,\beta} \) be lognormal under \( Q^{\alpha,\beta} \), given that

\[
S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}} \quad ?
\] (2)

Check distributions of \( S_{\alpha,\beta} \) under \( Q^{\alpha,\beta} \) for both LFM and LSM. Derive the LFM model under the LSM numeraire \( Q^{\alpha,\beta} \):

\[
dF_k(t) = \sigma_k(t) F_k(t) \left( \mu_k^{\alpha,\beta}(t) dt + dZ_k^{\alpha,\beta}(t) \right), \quad (3)
\]

\[
\mu_k^{\alpha,\beta} = \sum_{j=\alpha+1}^{\beta} (2(j \leq k) - 1) \tau_j P(t, T_j) \frac{\max(k,j)}{C_{\alpha,\beta}(t)} \sum_{i=\min(k+1,j+1)}^{\max(k,j)} \frac{\tau_i \rho_{k,i} \sigma_i F_i}{1 + \tau_i F_i}.
\]

When computing the swaption price as the \( Q^{\alpha,\beta} \) expectation

\[
C_{\alpha,\beta}(0) E^{\alpha,\beta}(S_{\alpha,\beta}(T_\alpha) - K)^+
\]

we can use either LFM (2,3) or LSM (1).

In general, \( S_{\alpha,\beta} \) coming from LSM (1) is LOGNORMAL, whereas \( S_{\alpha,\beta} \) coming from LFM (2,3) is NOT. But in practice...
**LFM instantaneous covariance structures**

LFM is natural for caps and LSM is natural for swaptions. **Choose.** We choose LFM and adapt it to price swaptions.

Recall: Under numeraire $P(\cdot, T_i) \neq P(\cdot, T_k)$:

$$
\frac{dF_k(t)}{F_k(t)} = \mu_k(t) dt + \sigma_k(t) F_k(t) dZ_k, \quad dZ_k dZ'_k = \rho \, dt
$$

Model specification: Choice of $\sigma_k(t)$ and of $\rho$.

- General Piecewise constant (GPC) vols, $\sigma_k(t) = \sigma_k,\beta(t)$

<table>
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<th>Inst. Vols</th>
<th>$t \in (0, T_0]$</th>
<th>$(T_0, T_1]$</th>
<th>$(T_1, T_2]$</th>
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Separable Piecewise const (SPC), $\sigma_k(t) = \Phi_k \psi_k(\beta(t) - 1)$

- Parametric Linear-Exponential (LE) vols

$$
\sigma_i(t) = \Phi_i \psi(T_{i-1} - t; a, b, c, d)
$$

$$
:= \Phi_i \left( [a(T_{i-1} - t) + d]e^{-b(T_{i-1} - t)} + c \right).
$$

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Caplet volatilities

Recall that under numeraire $P(\cdot, T_i)$:

$$dF_i(t) = \sigma_i(t) F_i(t) \, dZ_i, \quad dZ dZ' = \rho \, dt$$

Caplet: Strike rate $K$, Reset $T_{i-1}$, Payment $T_i$:

Payoff: $\tau_i(F_i(T_{i-1}) - K)^+$ at $T_i$.

"Call option" on $F_i$, $F_i \sim \text{lognormal under } Q_i$

$\Rightarrow$ Black’s formula, with Black vol. parameter

$$v_{T_{i-1}\text{-caplet}}^2 := \frac{1}{T_i - T_{i-1}} \int_0^{T_i - T_{i-1}} \sigma_i(t)^2 \, dt.$$

$v_{T_{i-1}\text{-caplet}}$ is $T_{i-1}$-caplet volatility

Only the $\sigma$’s have impact on caplet (and cap) prices, the $\rho$’s having no influence.
Caplet volatilities (cont’d)

\[
dF_i(t) = \sigma_i(t) F_i(t) \, dZ_i, \quad v^2_{T_{i-1}-}\text{caplet} := \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i(t)^2 \, dt.
\]

Under GPC vols, \( \sigma_k(t) = \sigma_{k,\beta}(t) \)

\[
v^2_{T_{i-1}-}\text{caplet} = \frac{1}{T_{i-1}} \sum_{j=1}^{i} (T_{j-1} - T_{j-2}) \sigma_{i,j}^2
\]

Under LE vols, \( \sigma_i(t) = \Phi_i \psi(T_{i-1} - t; a, b, c, d) \),

\[
T_{i-1} v^2_{T_{i-1}-}\text{caplet} = \Phi_i^2 I^2(T_{i-1}; a, b, c, d)
\]

\[
:= \Phi_i^2 \int_0^{T_{i-1}} \left( [a(T_{i-1} - t) + d] e^{-b(T_{i-1} - t)} + c \right)^2 \, dt.
\]
Term Structure of Caplet Volatilities

The term structure of volatility at time $T_j$ is a graph of expiry times $T_{h-1}$ against average volatilities $V(T_j, T_{h-1})$ of the related forward rates $F_h(t)$ up to that expiry time itself, i.e. for $t \in (T_j, T_{h-1})$.

Formally, at time $t = T_j$, graph of points
\[
\{(T_{j+1}, V(T_j, T_{j+1})), (T_{j+2}, V(T_j, T_{j+2})), \ldots, (T_{M-1}, V(T_j, T_{M-1}))\}
\]

\[
V^2(T_j, T_{h-1}) = \frac{1}{T_{h-1} - T_j} \int_{T_j}^{T_{h-1}} \sigma_h^2(t) dt, \quad h > j + 1.
\]

The term structure of vols at time 0 is given simply by caplets vols plotted against their expiries.

Different assumptions on the behaviour of instantaneous volatilities (SPC, LE, etc.) imply different evolutions for the term structure of volatilities in time as $t = T_0, t = T_1, t = T_2...$
The LIBOR and SWAP market model
Terminal and Instantaneous correlation

Swaptions depend on terminal correlations among fwd rates.

E.g., the swaption whose underlying is $S_{1,3}$ depends on

$$\text{corr}(F_2(T_1), F_3(T_1)).$$

This terminal corr. depends both on inst. corr. $\rho_{2,3}$

and and on the way the $T_2$ and $T_3$ caplet vols $v_2$ and $v_3$ are decomposed in instantaneous vols $\sigma_2(t)$ and $\sigma_3(t)$ for $t$ in $0, T_1$.

$$\text{corr}(F_2(T_1), F_3(T_1)) \approx \frac{\int_0^{T_1} \sigma_2(t)\sigma_3(t)\rho_{2,3}}{\sqrt{\int_0^{T_1} \sigma_2^2(t)dt} \sqrt{\int_0^{T_1} \sigma_3^2(t)dt}} =$$

$$= \rho_{2,3} \frac{\sigma_{2,1} \sigma_{3,1} + \sigma_{2,2} \sigma_{3,2}}{v_2 \sqrt{\sigma_{3,1}^2 + \sigma_{3,2}^2}}.$$

No such formula is available, in general, for short-rate models
\[ \text{corr}(F_2(T_1), F_3(T_1)) \approx \rho_{2,3} \frac{\sigma_{2,1}\sigma_{3,1} + \sigma_{2,2}\sigma_{3,2}}{v_2 \sqrt{\sigma_{3,1}^2 + \sigma_{3,2}^2}}. \]

Fix \( \rho_{2,3} = 1 \), \( \tau_i = 1 \) and caplet vols:

\[ v_2^2 = \sigma_{2,1}^2 + \sigma_{2,2}^2; \quad v_3^2 = \sigma_{3,1}^2 + \sigma_{3,2}^2 + \sigma_{3,3}^2. \]

Decompose \( v_2 \) and \( v_3 \) in two different ways: **First case**

\[ \sigma_{2,1} = v_2, \quad \sigma_{2,2} = 0; \quad \sigma_{3,1} = v_3, \quad \sigma_{3,2} = 0, \quad \sigma_{3,3} = 0. \]

In this case the above formula yields easily

\[ \text{corr}(F_2(T_1), F_3(T_1)) = \rho_{2,3} = 1. \]

The **second case** is obtained as

\[ \sigma_{2,1} = 0, \quad \sigma_{2,2} = v_2; \quad \sigma_{3,1} = v_3, \quad \sigma_{3,2} = 0, \quad \sigma_{3,3} = 0. \]

In this second case the above formula yields immediately

\[ \text{corr}(F_2(T_1), F_3(T_1)) = 0 \rho_{2,3} = 0. \]
Terminal and Instantaneous correlation

Swaptions depend on terminal correlation among forward rates ($\rho$'s and $\sigma$'s).

Instant. correl: Approximate $\rho \ (M \times M, \text{Rank } M)$ with a
$n$-rank $\rho^B = B \times B'$, with $B$ an $M \times n$ matrix, $n << M$.

$$dZ \ dZ' = \rho \ dt \quad \longrightarrow \quad B \ dW (B \ dW)' = BB' dt \ .$$

$$\rho^B = B \times B', \text{ with } B \text{ an } M \times n \text{ matrix, } n << M.$$  

A parametric form has to be chosen for $B$. Rebonato:

$$b_{i,1} = \cos \theta_{i,1}$$
$$b_{i,k} = \cos \theta_{i,k} \sin \theta_{i,1} \cdots \sin \theta_{i,k-1}, \ 1 < k < n,$$
$$b_{i,n} = \sin \theta_{i,1} \cdots \sin \theta_{i,n-1}, \text{ for } i = 1, 2, \ldots, M.$$

For $n = 2$, $\rho^B_{i,j} = b_{i,1}b_{j,1} + b_{i,2}b_{j,2} = \cos(\theta_i - \theta_j)$.

This structure consists of $M$ parameters $\theta_1, \ldots, \theta_M$ obtained either by forcing the LFM model to recover market swaptions prices (market implied data),

or through historical estimation (time-series/econometrics).
Example: historically estimated true $\rho$

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Rank-2 approximation:

$$\theta^{(2)} = [1.2367 \ 1.2812 \ 1.3319 \ 1.3961 \ 1.4947 \ 1.6469 \ 1.7455 \ 1.8097 \ 1.8604 \ 1.9049].$$

The resulting optimal rank-2 matrix $\rho(\theta^{(2)})$ is

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Figure 1: Problems of low rank correlation: sigmoid shape
Monte Carlo pricing swaptions with LFM

$$\tilde{E} \left( D(0, T_\alpha) \left[ S_{\alpha,\beta}(T_\alpha) - K \right]^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \right) =$$

$$= P(0, T_\alpha) E^\alpha \left[ \left( S_{\alpha,\beta}(T_\alpha) - K \right)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \right].$$

Since

$$S_{\alpha,\beta}(T_\alpha) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1 + \tau_j F_j(T_\alpha)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1 + \tau_j F_j(T_\alpha)}}$$

the above expectation depends on the joint distrib. under $Q^\alpha$ of

$$F_{\alpha+1}(T_\alpha), F_{\alpha+2}(T_\alpha), \ldots, F_\beta(T_\alpha)$$

Recall the dynamics of forward rates under $Q^\alpha$:

$$dF_k(t) = \sigma_k(t) F_k(t) \sum_{j=\alpha+1}^{k} \rho_{k,j} \tau_j \frac{\sigma_j F_j}{1 + \tau_j F_j(t)} dt + \sigma_k(t) F_k(t) dZ_k,$$
\[
\tilde{E} \left( D(0, T_\alpha) \left( S_{\alpha,\beta}(T_\alpha) - K \right)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \right) = \\
= P(0, T_\alpha) E^\alpha \left[ (S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i) \right].
\]

Since \( S_{\alpha,\beta}(T_\alpha) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(T_\alpha)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(T_\alpha)}} \)

Milstein scheme for \( \ln F \):

\[
\ln F_k^{\Delta t} (t + \Delta t) = \ln F_k^{\Delta t} (t) + \sigma_k(t) \sum_{j=\alpha+1}^{k} \frac{\rho_{k,j} \tau_j \sigma_j(t) F_j^{\Delta t}}{1 + \tau_j F_j^{\Delta t}} \Delta t + \\
- \frac{\sigma_k^2(t)}{2} \Delta t + \sigma_k(t)(Z_k(t + \Delta t) - Z_k(t))
\]

leads to an approximation such that there exists a \( \delta_0 \) with

\[
E^\alpha \left\{ \left| \ln F_k^{\Delta t}(T_\alpha) - \ln F_k(T_\alpha) \right| \right\} \leq C(T_\alpha)(\Delta t)^1 \quad \text{for all } \Delta t \leq \delta_0
\]

where \( C(T_\alpha) > 0 \) is a constant (strong convergence of order 1).

\( (Z_k(t + \Delta t) - Z_k(t)) \) is GAUSSIAN and KNOWN, easy to simulate.
Analytical swaption prices with LFM

Approximated method to compute swaption prices with the LFM without resorting to Monte Carlo simulation.

This method is rather simple and its quality has been tested in Brace, Dun, and Barton (1999) and by ourselves.

Recall the LSM leading to Black’s formula for swaptions:

\[
d S_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t) \, dW_{t}^{\alpha,\beta}, \quad Q^{\alpha,\beta}.
\]

A crucial role is played by the Black swap volatility

\[
\int_{0}^{T_{\alpha}} \sigma_{\alpha,\beta}^{2}(t) \, dt = \int_{0}^{T_{\alpha}} (d \ln S_{\alpha,\beta}(t))(d \ln S_{\alpha,\beta}(t))
\]

We compute an analogous approximated quantity in the LFM.

\[
S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_{i}(t) \, F_{i}(t),
\]

\[
w_{i}(t) = w_{i}(F_{\alpha+1}(t), F_{\alpha+2}(t), \ldots, F_{\beta}(t)) = \frac{\tau_{i} \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_{j} F_{j}(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_{k} \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_{j} F_{j}(t)}}.
\]
Freeze the $w$'s at time 0:

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) F_i(t) .$$

(variability of the $w$'s is much smaller than variability of $F$'s)

$$dS_{\alpha,\beta} \approx \sum_{i=\alpha+1}^{\beta} w_i(0) dF_i = (\ldots) dt + \sum_{i=\alpha+1}^{\beta} w_i(0) \sigma_i(t) F_i(t) dZ_i(t) ,$$

under any of the forward adjusted measures. Compute

$$dS_{\alpha,\beta}(t) dS_{\alpha,\beta}(t) \approx \sum_{i,j=\alpha+1}^{\beta} w_i(0) w_j(0) F_i(t) F_j(t) \rho_{i,j} \sigma_i(t) \sigma_j(t) dt .$$

The percentage quadratic covariation is $(d \ln S_{\alpha,\beta}(t))(d \ln S_{\alpha,\beta}(t)) \approx$

$$\approx \sum_{i,j=\alpha+1}^{\beta} w_i(0) w_j(0) F_i(t) F_j(t) \rho_{i,j} \sigma_i(t) \sigma_j(t) \frac{1}{S_{\alpha,\beta}(t)^2} dt .$$

Introduce a further approx by freezing again all $F$'s (as was done earlier for the $w$'s) to time zero: $(d \ln S_{\alpha,\beta})(d \ln S_{\alpha,\beta}) \approx$

$$\approx \sum_{i,j=\alpha+1}^{\beta} w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j} \frac{1}{S_{\alpha,\beta}(0)^2} \sigma_i(t) \sigma_j(t) dt .$$
Now compute the integrated percentage variance of $S$ as

$$(\text{Rebonato's Formula})$$

$$(v_{\alpha,\beta}^{\text{LFM}})^2 = \int_0^{T_\alpha} (d \ln S_{\alpha,\beta}(t)) (d \ln S_{\alpha,\beta}(t))$$

$$= \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t) \, dt .$$

$v_{\alpha,\beta}^{\text{LFM}}$ can be used as a proxy for the Black volatility $v_{\alpha,\beta}(T_\alpha)$.

Use Black’s formula for swaptions with volatility $v_{\alpha,\beta}^{\text{LFM}}$ to price swaptions \textbf{analytically} with the LFM.

It turns out that the approximation is not at all bad, as pointed out by Brace, Dun and Barton (1999) and by ourselves.

A slightly more sophisticated version of this procedure has been pointed out for example by Hull and White (1999).

This pricing formula is \textbf{ALGEBRAIC} and very quick (compare with short-rate models)

H–W refine this formula by differentiating $S_{\alpha,\beta}(t)$ without immediately freezing the $w$. Same accuracy in practice.
Analytical terminal correlation

By similar arguments (freezing the drift and collapsing all measures) we may find a formula for terminal correlation.

\[
\text{Corr}(F_i(T_\alpha), F_j(T_\alpha)) \text{ should be computed with } \text{MC simulation and depends on the chosen numeraire}
\]

Useful to have a first idea on the stability of the model correlation at future times.

Traders need to check this quickly, no time for MC

In Brigo and Mercurio (2001), we obtain easily

\[
\frac{\exp \left( \int_0^{T_\alpha} \sigma_i(t) \sigma_j(t) \rho_{i,j} \, dt \right) - 1}{\sqrt{\exp \left( \int_0^{T_\alpha} \sigma_i^2(t) \, dt \right) - 1} \sqrt{\exp \left( \int_0^{T_\alpha} \sigma_j^2(t) \, dt \right) - 1}} 
\]

\[
\approx \rho_{i,j} \frac{\int_0^{T_\alpha} \sigma_i(t) \sigma_j(t) \, dt}{\sqrt{\int_0^{T_\alpha} \sigma_i^2(t) \, dt} \sqrt{\int_0^{T_\alpha} \sigma_j^2(t) \, dt}},
\]

the second approximation as from Rebonato (1999). Schwartz’s inequality: terminal correlations are always smaller, in absolute value, than instantaneous correlations.
Calibration to swaptions prices

Swaption calibration: Find $\sigma$ and $\rho$ in LFM such that the LFM reproduces market swaption vols

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Table 1: Black vols of EURO ATM swaptions May 16, 2000

Table (brokers) not updated uniformly. Some entries may refer to older market situations.

“Temporal misalignment/Stale data”

Calibrated parameters $\sigma$ or $\rho$ might reflect this by weird configurations. If so:

Trust the model $\Rightarrow$ detect misalignments

Trust the data $\Rightarrow$ need a better parameterization.
Joint calibration to caps and swaptions

CALIBRATION: Need to find $\sigma(t)$ and $\rho$ such that the market prices of caps and swaptions are recovered by $\text{LFM}(\sigma, \rho)$.

$\text{caplet-volat-LFM}(\sigma) = \text{market-caplet-volat}$ (Almost automatic).

$\text{swaptions-LFM}(\sigma, \rho) = \text{market-swaptions}$.

Caplets: Algebraic formula; Immediate calibration, almost automatic.

Swaptions: In principle Monte Carlo pricing. But MC pricing at each optimization step is too computationally intensive.

Use Rebonato’s approximation and at each optimization step evaluate swaptions analytically with the LFM model.
Joint calibration: Market cases

SPC vols, $\sigma_k(t) = \sigma_{k,\beta(t)} := \Phi_k \psi_{k-(\beta(t)-1)}$.

$\rho$ rank-2 with angles $-\pi/2 < \theta_i - \theta_{i-1} < \pi/2$

Data below as of May 16, 2000, $F(0; 0, 1y) = 0.0452$, plus swaptions matrix as in the earlier slide.

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<td>0.0961</td>
<td>1.5708</td>
</tr>
<tr>
<td>18</td>
<td>0.0000</td>
<td>0.0986</td>
<td>1.5708</td>
</tr>
<tr>
<td>19</td>
<td>0.6551</td>
<td>0.1004</td>
<td>1.5708</td>
</tr>
</tbody>
</table>
Joint calibration: Market cases (cont’d)

Quality of calibration: Caplets are fitted exactly, whereas we calibrated the whole swaptions volatility matrix except for the first column.

Matrix: \( \frac{100(\text{Mkt swaptions vol} - \text{LFM swaption vol})}{\text{Mkt swaptions vol}} \):

<table>
<thead>
<tr>
<th></th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
<th>6y</th>
<th>7y</th>
<th>8y</th>
<th>9y</th>
<th>10y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1y</td>
<td>-0.24</td>
<td>1.29%</td>
<td>0.61</td>
<td>-0.59</td>
<td>-0.28</td>
<td>0.97</td>
<td>-0.64</td>
<td>1.02</td>
<td>-0.87</td>
</tr>
<tr>
<td>2y</td>
<td>-1.65</td>
<td>-1.29</td>
<td>-1.09</td>
<td>-0.33</td>
<td>0.03</td>
<td>-0.61</td>
<td>-0.37</td>
<td>-1.04</td>
<td>-0.89</td>
</tr>
<tr>
<td>3y</td>
<td>1.03</td>
<td>1.11</td>
<td>0.51</td>
<td>1.45</td>
<td>0.79</td>
<td>1.08</td>
<td>1.03</td>
<td>1.30</td>
<td>0.79</td>
</tr>
<tr>
<td>4y</td>
<td>0.13</td>
<td>-1.05</td>
<td>-0.80</td>
<td>-0.29</td>
<td>-0.33</td>
<td>-0.16</td>
<td>0.49</td>
<td>0.02</td>
<td>0.23</td>
</tr>
<tr>
<td>5y</td>
<td>0.89</td>
<td>0.07</td>
<td>-0.09</td>
<td>-0.16</td>
<td>-1.27</td>
<td>-0.50</td>
<td>0.00</td>
<td>-0.80</td>
<td>0.37</td>
</tr>
<tr>
<td>7y</td>
<td>1.15</td>
<td>0.53</td>
<td>0.59</td>
<td>0.12</td>
<td>-0.33</td>
<td>-0.66</td>
<td>-0.58</td>
<td>0.10</td>
<td>0.93</td>
</tr>
<tr>
<td>10y</td>
<td>-1.23</td>
<td>-0.61</td>
<td>0.64</td>
<td>0.07</td>
<td>0.46</td>
<td>0.45</td>
<td>-0.37</td>
<td>-0.64</td>
<td>-0.20</td>
</tr>
</tbody>
</table>

Calibr error OK for 19 caplets and 63 swaptions, but... calibrated \( \theta \)'s imply erratic, oscillating (\( +/- \)) \( \rho \)'s and 10y terminal correlations:

<table>
<thead>
<tr>
<th></th>
<th>10y</th>
<th>11y</th>
<th>12y</th>
<th>13y</th>
<th>14y</th>
<th>15y</th>
<th>16y</th>
<th>17y</th>
<th>18y</th>
</tr>
</thead>
<tbody>
<tr>
<td>10y</td>
<td>1.000</td>
<td>0.574</td>
<td>0.625</td>
<td>0.525</td>
<td>0.602</td>
<td>0.500</td>
<td>0.505</td>
<td>0.445</td>
<td>0.418</td>
</tr>
<tr>
<td>11y</td>
<td>0.574</td>
<td>1.000</td>
<td>0.801</td>
<td>0.976</td>
<td>0.805</td>
<td>0.926</td>
<td>0.750</td>
<td>0.760</td>
<td>0.646</td>
</tr>
<tr>
<td>12y</td>
<td>0.625</td>
<td>0.801</td>
<td>1.000</td>
<td>0.712</td>
<td>0.917</td>
<td>0.653</td>
<td>0.745</td>
<td>0.590</td>
<td>0.588</td>
</tr>
<tr>
<td>13y</td>
<td>0.525</td>
<td>0.976</td>
<td>0.712</td>
<td>1.000</td>
<td>0.697</td>
<td>0.906</td>
<td>0.622</td>
<td>0.721</td>
<td>0.550</td>
</tr>
<tr>
<td>14y</td>
<td>0.602</td>
<td>0.805</td>
<td>0.917</td>
<td>0.697</td>
<td>1.000</td>
<td>0.655</td>
<td>0.866</td>
<td>0.570</td>
<td>0.666</td>
</tr>
<tr>
<td>15y</td>
<td>0.500</td>
<td>0.926</td>
<td>0.653</td>
<td>0.906</td>
<td>0.655</td>
<td>1.000</td>
<td>0.627</td>
<td>0.855</td>
<td>0.524</td>
</tr>
<tr>
<td>16y</td>
<td>0.505</td>
<td>0.750</td>
<td>0.745</td>
<td>0.622</td>
<td>0.866</td>
<td>0.627</td>
<td>1.000</td>
<td>0.597</td>
<td>0.846</td>
</tr>
<tr>
<td>17y</td>
<td>0.445</td>
<td>0.760</td>
<td>0.590</td>
<td>0.721</td>
<td>0.570</td>
<td>0.855</td>
<td>0.597</td>
<td>1.000</td>
<td>0.556</td>
</tr>
<tr>
<td>18y</td>
<td>0.418</td>
<td>0.646</td>
<td>0.588</td>
<td>0.550</td>
<td>0.666</td>
<td>0.524</td>
<td>0.846</td>
<td>0.556</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Evolution of the term structure of caplet volatilities
Loses the "humped shape" after a short time.
Becomes somehow "noisy"
"previous results on fitted correlation: future market structures implied by the fitted model are not regular under SPC."
Tried other calibrations with SPC $\sigma$’s

Tried: More stringent constraints on the $\theta$

Fixed $\theta$ both to typical and atypical values, leaving the calibration only to the vol parameters

Fixed $\theta$ so as to have all $\rho = 1$.

Summary: To have good calibration to swaptions need to keep the angles unconstrained and allow for partly oscillating $\rho$’s.

If we force “smooth/monotonic” $\rho$’s and leave calibr to vols, results are essentially the same as in the case of a one-factor LFM with $\rho = 1$.

Maybe inst correlations do not have a strong link with European swaptions prices? (Rebonato)

Maybe permanence of “bad results”, no matter the particular “smooth” choice of fixed $\rho$, reflects an impossibility of a low-rank $\rho$ to decorrelate quickly fwd rates in a steep initial pattern? (Rebonato)

3-factor $\rho$’s do not help. Obvious remedy would be increasing drastically $\#$ factors. But MC...
Joint calibration: Market cases (cont’d)

Calibration with the LE parametric $\sigma$’s.

Same inputs as before

Rank-2 $\rho$ with $-\pi/3 < \theta_i - \theta_{i-1} < \pi/3$, $0 < \theta_i < \pi$

Constraint “$1 - 0.1 \leq \Phi_i(a, b, c, d) \leq 1 + 0.1$”

Calibrated parameters and calibration error (caps exact):

$a = 0.29342753$, $b = 1.25080230$, $c = 0.13145869$, $d = 0.00$

$\theta_{1\div7} = [1.75411 \ 0.57781 \ 1.68501 \ 0.58176 \ 1.53824 \ 2.43632 \ 0.88011]$

$\theta_{8\div12} = [1.89645 \ 0.48605 \ 1.28020 \ 2.44031 \ 0.94480]$

$\theta_{13\div19} = [1.34053 \ 2.91133 \ 1.99622 \ 0.70042 \ 0.81518 \ 2.38376]$

<table>
<thead>
<tr>
<th></th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
<th>6y</th>
<th>7y</th>
<th>8y</th>
<th>9y</th>
<th>10y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1y</td>
<td>2.28</td>
<td>-3.74</td>
<td>-3.19</td>
<td>-4.68</td>
<td>2.46</td>
<td>1.50</td>
<td>0.72</td>
<td>1.33</td>
<td>-1.42</td>
</tr>
<tr>
<td>2y</td>
<td>-1.23</td>
<td>-7.67</td>
<td>-9.97</td>
<td>2.10</td>
<td>0.49</td>
<td>1.33</td>
<td>1.56</td>
<td>-0.44</td>
<td>1.88</td>
</tr>
<tr>
<td>3y</td>
<td>2.23</td>
<td>-6.20</td>
<td>-1.30</td>
<td>-1.32</td>
<td>-1.43</td>
<td>1.86</td>
<td>-0.19</td>
<td>2.42</td>
<td>1.17</td>
</tr>
<tr>
<td>4y</td>
<td>-2.59</td>
<td>9.02</td>
<td>1.70</td>
<td>0.79</td>
<td>3.22</td>
<td>1.19</td>
<td>4.85</td>
<td>3.75</td>
<td>1.21</td>
</tr>
<tr>
<td>5y</td>
<td>-3.26</td>
<td>-0.28</td>
<td>-8.16</td>
<td>-0.81</td>
<td>-3.56</td>
<td>-0.23</td>
<td>-0.08</td>
<td>-2.63</td>
<td>2.62</td>
</tr>
<tr>
<td>7y</td>
<td>0.10</td>
<td>-2.59</td>
<td>-10.85</td>
<td>-2.00</td>
<td>-3.67</td>
<td>-6.84</td>
<td>2.15</td>
<td>1.19</td>
<td>0.00</td>
</tr>
<tr>
<td>10y</td>
<td>0.29</td>
<td>-3.44</td>
<td>-11.83</td>
<td>-1.31</td>
<td>-4.69</td>
<td>-2.60</td>
<td>4.07</td>
<td>1.11</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Inst correlations are again oscillating and non-monotonic.
Terminal correlations share part of this negative behaviour.
Joint calibration: Market cases (cont’d)

Evolution of term structure of vols looks better

Many more experiments with rank-three correlations, less or more stringent constraints on the angles and on the $\Phi$’s.

Fitting to the whole swaption matrix can be improved, but at the cost of an erratic behaviour of both correlations and of the evolution of the term structure of volatilities in time.

3-factor choice does not seem to help that much, as before. LE $\sigma$’s allow for an easier control of the evolution of the term structure of vols, but produce more erratic $\rho$’s: most of the “noise” in the swaption data ends up in the angles (we have only 4 vol parameters $a$, $b$, $c$, $d$ for fitting swaptions)
Calibration with GPC vols: one to one

\( \rho \)'s exogenously given (e.g. historical estimation)

\[
(\nu^{\text{LFM}}_{\alpha,\beta})^2 \approx \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^\beta \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t) \, dt,
\]

\[
T_\alpha S_{\alpha,\beta}(0)^2 v_{\alpha,\beta}^2 = \\
= \sum_{i,j=\alpha+1}^{\beta-1} w_i w_j F_i F_j \rho_{i,j} \sum_{h=0}^\alpha (T_h - T_{h-1}) \sigma_{i,h+1} \sigma_{j,h+1} \\
+ 2 \sum_{j=\alpha+1}^{\beta-1} w_\beta w_j F_\beta F_j \rho_{\beta,j} \sum_{h=0}^{\alpha-1} (T_h - T_{h-1}) \sigma_{\beta,h+1} \sigma_{j,h+1} \\
+ 2 \sum_{j=\alpha+1}^{\beta-1} w_\beta w_j F_\beta F_j \rho_{\beta,j} (T_\alpha - T_{\alpha-1}) \boxed{\sigma_{\beta,\alpha+1}} \sigma_{j,\alpha+1} \\
+ w_\beta^2 F_\beta^2 \sum_{h=0}^{\alpha-1} (T_h - T_{h-1}) \sigma_{\beta,h+1}^2 \\
+ w_\beta^2 F_\beta^2 (T_\alpha - T_{\alpha-1}) \boxed{\sigma_{\beta,\alpha+1}^2}.
\]

Solve this 2nd order eq: all quantities known or previously calculated except \( \sigma_{\beta,\alpha+1} \), provided that the “upper diagonal part” of the input swaption matrix is visited left to right and top down, starting from the upper left corner \( \nu_{0,1} = \sigma_{1,1} \).

The LIBOR and SWAP market model
Calibr with general PC vols: One to one corresp with swaption vols (cont’d)

<table>
<thead>
<tr>
<th>Length Maturity</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0 = 1y$</td>
<td>$v_{0,1}$, $\sigma_{1,1}$</td>
<td>$v_{0,2}$, $\sigma_{1,1}, \sigma_{2,1}$</td>
<td>$v_{0,3}$, $\sigma_{1,1}, \sigma_{2,1}, \sigma_{3,1}$</td>
</tr>
<tr>
<td>$T_1 = 2y$</td>
<td>$v_{1,2}$, $\sigma_{2,1}, \sigma_{2,2}$</td>
<td>$v_{1,3}$, $\sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}, \sigma_{3,2}$</td>
<td>-</td>
</tr>
<tr>
<td>$T_2 = 3y$</td>
<td>$v_{2,3}$, $\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}$</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Problem: can obtain negative or imaginary $\sigma$’s.

Possible cause: Illiquidity/stale data on the $v$’s.

Possible remedy: Smooth the input swaption $v$’s matrix with a 17-dimensional parametric form and recalibrate: imaginary and negative vols $\sigma$ disappear.

Term structure of caplet vols evolves regularly but loses hump

Instantaneous correlations good because chosen exogenously

Terminal correlations positive and monotonically decreasing

This form can help in Vega breakdown analysis
Conclusions

Some desired calibration features:

- A small rank for $\rho$ in view of Monte Carlo
- A small calibration error;
- Positive and decreasing instantaneous correlations;
- Positive and decreasing terminal correlations;
- Smooth and stable evolution of the term structure of vols;

Can achieve these targets through a low $\#$ of factors? Difficult...

Try and combine many of the ideas presented here
The one-to-one formulation is perhaps the most promising:
Fitting to swaptions is exact; can fit caps by introducing infra-correlations; instantaneous correlation OK by construction; Terminal correlation not spoiled by the fitted $\sigma$'s; Terms structure evolution smooth but not fully satisfactory qualitatively.

**Requirements hardly checkable with short-rate models**

More mathematically-advanced issues: Smile modeling:

$$dF_k = \nu_k(t, F_k) F_k dZ_k,$$

functional forms for $\nu_k$ leading to caplet prices that are linear combinations of Black prices.


http://www.damianobrigo.it/book.html