Modelling Interest Rates with Lévy Processes

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Modelling Interest Rates with Lévy Processes

Introduction:
Modelling stock returns and interest rates

Lévy Processes

Lévy Processes and Interest Rate Models

The Random Lattice

Numerical Results
Modelling Interest Rates with Lévy Processes

Introduction:
Modelling stock returns and interest rates
Empirical Regularities

Want to model stock returns and interest rates.

What should a model provide?
  1) Fat tailed distributions
  2) Infinitely divisible distributions
  3) Finite moments
  4) Multivariate extensions
  5) …Tractability…

Almost all literature is on asset return distributions. Relatively little in interest rates:
  Lots of research waiting to happen…
Interest Rate Models

Interest rate models,
    typically driven by Wiener processes.
Vasicek: \[ dr_t = \alpha(\mu - r_t)dt + \sigma dz_t. \]

Problems:
    Can’t match volatility smiles.
    Can’t account for empirical stochastic volatility.
    Empirically, \( r_t \) jumps.

Generalise: incorporate jumps?
\[ dr_t = \alpha(\mu - r_t)dt + \sigma dz_t + J_t dN_t, \]
where \( N_t \) is a Poisson process, intensity \( \lambda_t \)
    \( J_t \) is a random jump size.

But: How to specify \( J_t \)?
    Seems arbitrary, how to provide a context?
Solution?

Replace

$z_t$, Wiener process, by
$L_t$, Lévy process.

Lévy processes
Generalise Wiener processes.
Theoretically tractable
Very flexible.
(Contain compound Poisson)
Potential to account for smiles, etc
What does a stochastic process look like?

Probability measure on state space $\Omega$, 
$\Omega = \{ f : \mathbb{R}^+ \rightarrow \mathbb{R} \}$

$\Omega$ is huge.
Almost every $f$ is totally discontinuous, 
eg, different on $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$

Space of continuous sample paths
$\Omega^c = \{ f : \mathbb{R}^+ \rightarrow \mathbb{R} | f \text{ cts} \} \subset \Omega$.
Sample space for Brownian motions

Tiny subset of $\Omega$. 
Lévy processes

Lévy Processes:
Stationary with independent increments,
Continuous in probability,
(starts at zero, càdlàg).

Convergence in probability:
∀ ε > 0, Pr[ |X_t - X_s| > ε ] → 0 as s → t
If not true then numerical methods might be hard…

Example:
if t ∈ Q, then Pr[ X_t = 0] = ½,
Pr[ X_t = 2] = ½,
if t ∉ Q, then Pr[ X_t = 0] = ½,
Pr[ X_t = 1] = ½,
Not convergent in probability.

As ∆t → 0 then p_{l-k} → 0 for l - k large (∆x constant).
Modelling Price and Rate Movements

Different approaches:

1) Write down the SDEs.
   Stochastic volatility + jumps?

2) Specify the conditional distributions:
   i) Give the distribution itself, $F(S_t \mid S_0)$
   ii) Give the density, $f(S_t \mid S_0)$, if it exists
   iii) Give the inverse distribution, $F^{-1}(S_t \mid S_0)$

3) Specify the process as a time change

4) Specify as a Lévy process,
   with its Lévy measure, $\nu(dx)$.
   May have a Lévy density $k(x)$, $\nu(dx) = k(x)dx$.
   Approximate as compound Poisson?

5) Specify the process by its time copula.

Specify by backing out from prices?
Specify a functional form, then fit to prices?
Lévy Processes and No-arbitrage Pricing

If there is no arbitrage:

Price processes can be transformed into Martingales under the pricing measure.

A semimartingale can be transformed to a Martingale under some equivalent measure.

No-arbitrage?

Asset price processes, rates, etc, must be semimartingales.

A Lévy process is a semimartingale:

\[ X_t \text{ a Lévy process then } X_t = Y_t + Z_t \text{ where} \]

\[ Y_t \text{ and } Z_t \text{ are Lévy}, \]

\[ Y_t \text{ is a martingale with bounded jumps,} \]

\[ Z_t \text{ has FV paths on compact intervals.} \]
Modelling Stock Returns with Lévy Process

Stock returns are fat tailed.
Model with
Stochastic volatility?
Jumps?

Assume stock return process (under the EMM) is
\[ S_t = S_0 \exp( rt + \lambda_t - wt) \]
where
\( \lambda_t \) is a Lévy process,
wt term makes \( S_t e^{-rt} \) a martingale, \( e^w = E[\exp(\lambda_1)] \).

Lots of literature:
Empirical fits to time series, volatility smile.
Formulae for (European) options.
...Some on pricing methods.

Interest Rates? Very little literature as yet.

Problem:
In general, hard to price non-vanilla derivatives.
How to value Bermudan/American puts?
Numerical Methods

Need for numerical solutions:
  Few formulae for non-vanilla options.
  Valuing American and Bermudan options?

Existing methods for option pricing:
  Explicit solutions
  Fourier transform methods
  Monte Carlo:
    Directly.
    As subordinated Brownian motion,
    Mean-variance mixture.
    As compound Poisson approximation.
  ‘Method of Lines’

Lattice methods:
  Can price American and Bermudan options.
  Flexible for different payoffs.
  Can get high accuracy cheaply.
Modelling Interest Rates with Lévy Processes

Lévy Processes

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Lévy Processes: Definition and Examples

Lévy Processes:
    Stationary with independent increments,
    Continuous in probability,
    (starts at zero, càdlàg).

Lévy processes applied in finance:
    Generalised hyperbolic processes (GH),
    Normal inverse Gaussian (NIG),
    Variance Gamma (VG).

VG and NIG are special cases of GH,
but are unique subclasses closed under convolution,
therefore worth considering separately.

Have explicit functions for the densities
of these processes.

Lévy processes are infinitely divisible.
Infinitely Divisible Measures and Processes

A measure $\mu$ on $\mathbb{R}^d$ is infinitely divisible if
\[ \forall n \text{ there exists } \mu_n \text{ such that } \mu = (\mu_n)^* = \mu_n^* \cdots ^* \mu_n \text{ (n-fold convolution)} \]

If $\mu$ is infinitely divisible then $\hat{\mu}(z) = (\hat{\mu}_n(z))^n$

Infinitely divisible distributions:
- Lévy processes.
- GIG processes.
- On $\mathbb{R}^d$: Gaussian, Cauchy
- On $\mathbb{R}$: Poisson, exponential, $\Gamma$.

Can’t be infinitely divisible if
- $\mu$ has bounded support,
- $\hat{\mu}$ has zeros.
Uniform distribution is not infinitely divisible.

Every infinitely divisible distribution is the limit of a sequence of compound Poisson distributions.
Infinitely Divisible Measures

\( \mu \) is infinitely divisible on \( \mathbb{R}^d \)
iff there exists a Lévy process (in law)
with \( F_{X_1} = X_1|X_0 = \mu \)

\( X_t \) is unique up to identity in law.
It has a càglàg modification.
The Lévy-Khintchine representation

If $X_t$ is infinitely divisible then

$$\hat{\mu}(z) = \exp(\phi(z)), \quad z \in \mathbb{R},$$

with

$$\phi(z) = -\frac{1}{2}z'Az + iz'\gamma$$

$$+ \int_{\mathbb{R}^d} (e^{iz'x} - 1 - iz'x.1_D(x))\nu(dx).$$

where

- $A$ is a symmetric non-negative definite matrix,
- $\gamma \in \mathbb{R}^d$,
- $\nu$ is a measure on $\mathbb{R}^d$, such that
  $$\nu\{0\} = 0,$$
  $$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty,$$
- $D = \{ x \mid |x| \leq 1 \}$ is the unit ball in $\mathbb{R}^d$,
- $\nu$ is not necessarily a probability measure.

Need not be integrable.

i) $(A,\nu,\gamma)$ is unique

ii) All $(A,\nu,\gamma)$ give infinitely divisible distributions

$(A,\nu,\gamma)$ is the generating triplet of $\mu$. 
Notes:

If $\mu$ has generating triplet $(A, \nu, \gamma)$ then $\mu^t$ has generating triplet $(tA, t\nu, t\gamma)$.

$\nu$ is called the Lévy measure of $\mu$.
If $\nu(dx) = k(x)dx$ has a density,

$k$ is called the Lévy density of $\mu$.

The Lévy-Khintchine representation is not unique. Can have:

$$\phi(z) = -\frac{1}{2}z'Az + iz'\gamma_c$$
$$+ \int_{\mathbb{R}^d} (e^{iz'x} - 1 - iz'x.c(x))\nu(dx).$$

where, eg,

$$c(x) = (1 + |x|^2)^{-1},$$

$$c(x) = 1_{\{|x| \leq \varepsilon\}}(x), \ \varepsilon > 0, \ \text{etc}$$

when

$$\gamma_c = \gamma + \int_{\mathbb{R}^d} x( c(x) - 1_D(x) )\nu(dx).$$

Write $(A, \nu, \gamma_c)_c$.
Centre and Drift

Suppose that $\int_{|x| \leq 1} |x|^2 \nu(dx) < \infty$,
then can set $c(x) = 0$ and
$\phi(z) = -\frac{1}{2} z' A z + i z' \gamma_0 + \int_{\mathbb{R}^d} (e^{i z' x} - 1) \nu(dx)$.
This $\gamma_0$ is the drift of $\mu$.

Suppose that $\int_{|x| > 1} |x|^2 \nu(dx) < \infty$,
then can set $c(x) = 1$ and
$\phi(z) = -\frac{1}{2} z' A z + i z' \gamma_1 + \int_{\mathbb{R}^d} (e^{i z' x} - 1 - i z' x) \nu(dx)$.
This $\gamma_1$ is the centre of $\mu$.

If $\gamma_1$ exists then $\gamma_1 = \int_{\mathbb{R}^d} x \mu(dx)$ is the mean of $\mu$.

$\mu$ Gaussian then $\nu = 0$ and $\gamma_0 = \gamma_1$.
Brownian motion with drift,
$\gamma_0$ is the drift of the Brownian motion.
Observations
\(\mu\) compound Poisson then
\[A = 0, \quad \nu = c\sigma, \quad \gamma_0 = 0.\]
Jump times are exponential, mean \(c\), each jump size is distributed as \(cv\).

\(\Gamma\)-distribution, parameters \(c, \alpha > 0\), then
\[\phi(z) = c\int_{[0,\infty)}(e^{ixz} - 1)e^{-\alpha x}dx,\]
so \(A = 0, \quad \nu(dx) = c\frac{e^{-\alpha x}}{x}1_{[0,\infty)}(x)dx, \quad \gamma_0 = 0.\)
This \(\nu\) has infinite mass.

\(X_t\) additive, continuous sample paths as
iff \(X_t\) has Gaussian distribution \(\forall t,\)
ie, \(X_t\) is Brownian motion.

\(A\) is Gaussian covariance of \(\mu\).
\(\nu = 0\) iff \(\mu\) is Gaussian.

\(A = 0,\) then is purely non-Gaussian
\(A, \gamma = 0,\) then \(\mu\) is pure jump.
Connections

$X_t$ additive process (in law) then, for all $t$, $F_{X_t}$ is infinitely divisible.

$(A_t, \nu_t, \gamma_t)$ triple for $\mu_t = F_{X_t}$ then $X_t$ is additive (in law) iff $(A_0, \nu_0, \gamma_0) = (0, 0, 0)$

\[
\begin{align*}
&\text{for } 0 \leq s \leq t < \infty, \quad z'A_s z \leq z'A_t z, \quad \nu_s(B) \leq \nu_t(B), \\
&\text{as } s \to t, \quad z'A_s z \to z'A_t z, \quad \nu_s(B) \to \nu_t(B), \quad \gamma_s \to \gamma_t.
\end{align*}
\]

Effectively:

- Infinitely divisible distribution $\iff$ Lévy process $\iff$ Generating triple $(A, \nu, \gamma)$

$X_t$ a Lévy process on $\mathbb{R}^d$, generating triple $(A, \nu, \gamma)$.

Is type A: if $A = 0$, $\nu(\mathbb{R}^d) < \infty$,

- type B: if $A = 0$, $\nu(\mathbb{R}^d) = \infty$, $\int_{|x| \leq 1} |x| \nu(dx) < \infty$,

- type C: if $A \neq 0$, or $\int_{|x| \leq 1} |x| \nu(dx) = \infty$. 
Sample path properties

Sample paths of $X_t$ are:

cts iff $\nu = 0$,

Piecewise constant iff

i) $X_t$ is type A with $\gamma_0 = 0$, or

ii) $X_t$ is compound Poisson

$\nu(R^d) = \infty$, then

jump times are countable, dense in $[0,\infty)$.  

$0 < \nu(R^d) < \infty$, then

jump times are countable, but not dense as.  

Time to first jump is exponential, mean $\nu(R^d)^{-1}$.

$X_t$ is type A or B

then has finite variation on $(0,t]$, as.

$X_t$ is type C

then has infinite variation on $(0,t]$, $\forall t$.  

The Generalised Hyperbolic Distribution

(Barndorff-Nielsen (01), Eberlein (01), Rydberg (99))

The density is:

\[ f_{GH}(x|\lambda, \alpha, \beta, \delta, \mu) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi \alpha^{\lambda-1/2}} \delta^\lambda K_\lambda \left( \delta \sqrt{\alpha^2 - \beta^2} \right) \cdot (\delta^2 + (x-\mu)^2)^{(\lambda-1/2)/2} \times K_{\lambda-1/2}(\alpha(\delta^2 + (x-\mu)^2)^{-1/2}) \exp(\beta(x-\mu)) , \]

where \( K_\nu(z) = \frac{1}{2} \int_0^\infty y^{\nu-1} \exp\left(-\frac{1}{2} z(y + y^{-1})\right) dy \)

is the modified Bessel function of the third kind.

Parameters: \( \alpha > 0, \) shape,
\( 0 \leq \beta < \alpha, \) skewness,
\( \lambda \in \mathbb{R}, \) class of the distribution,
\( \mu \in \mathbb{R}, \) location,
\( \delta > 0, \) scale,

Reparameterise: replace \( \alpha, \beta \) by

\[ \xi = \left(1 + \delta \sqrt{\alpha^2 - \beta^2}\right)^{-1/2}, \chi = \xi \beta / \alpha, \] so \( 0 \leq |\chi| < \xi < 1. \)

\( \xi \) and \( \chi \) are invariant under \( X \rightarrow aX + b. \)
\( \lambda = 1: \) Hyperbolic distribution

Then \( K_{\frac{1}{2}}(z) = (\pi/2z)^{\frac{1}{2}}e^{-z} \), and

\[
f_H(x) = f_H(x|\alpha, \beta, \delta, \mu) = f_{GH}(x|1, \alpha, \beta, \delta, \mu)
= \frac{(\alpha^2-\beta^2)^{\frac{1}{2}}}{\alpha \delta K_1(\delta \sqrt{\alpha^2-\beta^2})} \exp(-\alpha(\delta^2+(x-\mu)^2)^{\frac{1}{2}} + \beta(x-\mu))
\]

Centred if \( \mu = 0 \), symmetric if \( \beta = 0 \).

Get special cases (\( \xi - \chi \) parameterisation):
- \( \xi \to 0 \), Normal
- \( \xi \to 1 \), Laplace
- \( \chi \to \pm \xi \), Generalised inverse Gaussian
- \( |\chi| \to 1 \), Exponential

\( \lambda = -\frac{1}{2}: \) Normal Inverse Gaussian distribution

\[
f_{NIG}(x|\alpha, \beta, \delta, \mu) = f_{GH}(x|-\frac{1}{2}, \alpha, \beta, \delta, \mu)
= \frac{\alpha \delta}{\pi} \exp(\delta \sqrt{\alpha^2-\beta^2} + \beta(x-\mu)) K_1(\alpha \sqrt{\delta^2+(x-\mu)^2}) \sqrt{\delta^2+(x-\mu)^2}
\]

Distribution of first hitting times of a 2-dim BM, starting at \((\mu, 0)\) to \(\mathbb{R} \times \{\delta\}\), drift \((\beta, \sqrt{\alpha^2-\beta^2})\), vol \((1, 1)\).
Representing GH distributions
Represent as mixtures with GIG distributions.

Generalised inverse Gaussian distribution

\[ f_{\text{GIG}}(x|\lambda, \delta, \gamma) = \frac{(\delta/\gamma)^{\lambda}}{2K_\lambda(\sqrt{\gamma} \delta)} \lambda^{-1} \exp\left(-\frac{1}{2} \left(\delta^2 x + \gamma^2 x^{-1}\right)\right) \]

\( x > 0, \quad \text{if } \lambda > 0 \text{ then } \delta \geq 0, \gamma > 0 \)
\( \quad \text{if } \lambda = 0 \text{ then } \delta > 0, \gamma > 0 \)
\( \quad \text{if } \lambda > 0 \text{ then } \delta > 0, \gamma \geq 0 \)

Reciprocal inverse Gaussian

\[ f_{\text{RIP}}(x | \delta, \gamma) = f_{\text{GIG}}(x | \frac{1}{2}, \delta, \gamma) \]

Inverse Gaussian distribution \((\lambda = -\frac{1}{2})\)

\[ f_{\text{IG}}(x | \delta, \gamma) = f_{\text{GIG}}(x | -\frac{1}{2}, \delta, \gamma) \]
\[ = \frac{\delta}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{\gamma^2}{2 \delta} \left(x - \frac{\delta}{\gamma}\right)^2\right), \quad x > 0. \]

Gamma \((\delta = 0)\)

\[ f_{\Gamma}(x | \lambda, \gamma) = f_{\text{GIG}}(x | \lambda, 0, \gamma) = \Gamma(\lambda, \frac{1}{2} \gamma^2) \]
\[ \Gamma(\nu, a) = a^\nu \frac{x^{\nu-1}}{\Gamma(\nu)} \exp(-ax), \quad \nu > 0. \]

Reciprocal Gamma \((\gamma = 0)\)

\[ f_{\Gamma^{-1}}(x | \lambda, \delta) = f_{\text{GIG}}(x | -\lambda, \delta, 0) = \Gamma^{-1}(\lambda, \frac{1}{2} \delta^2), \quad \lambda > 0. \]
Relationship to Mean-Variance Mixtures

Mean-Variance mixture:
location $\mu$, correlation matrix $\Delta$, drift $\beta$, mixing distribution $F$.
$\mu, \beta \in \mathbb{R}^d$, $\Delta$ symmetric positive definite, $|\Delta| = 1$.

Let $u \sim F$. Write $N$ for normal distribution function. Then mean-variance mixture $M(\mu, \beta, \Delta; F)$ has $M_{X|u} \sim N(\mu + u\beta, u\Delta)$.

Characteristic function of $M$:
$\hat{M}(z) = e^{iz^\prime \mu} \hat{F}(z^\prime \beta + \frac{1}{2}z^\prime \Delta z)$

Convolutions:
$M(\mu, \beta, \Delta, F)^n = M(n\mu, \beta, \Delta; F^n)$

eg, Generalised hyperbolic distribution:
$F_{GH}(\lambda, \alpha, \beta, \delta, \mu, \Delta) = M(\mu, \beta, \Delta; N^-(\lambda, \delta, \gamma))$
$\gamma^2 = \alpha^2 - \beta^\prime \Delta \beta$,
$N^-(\lambda, \delta, \gamma)$ is generalised inverse Gaussian.
Mixtures of Distributions

Write \( n(x|\mu, \Delta) \) for the normal density function. Then (\( d = 1 \))

\[
f_{GH}(x|\lambda, \alpha, \beta, \delta, \mu) = \int_{0}^{\infty} n(x|\mu+\beta u, u)f_{GIG}(u|\lambda, \delta, \sqrt{\alpha^2-\beta^2})du
\]

\[
f_{H}(x|\alpha, \beta, \delta, \mu) = \int_{0}^{\infty} n(x|\mu+\beta u, u)f_{IG}(u|\delta, \sqrt{\alpha^2-\beta^2})du
\]

\[
f_{NIG}(x|\alpha, \beta, \delta, \mu) = \int_{0}^{\infty} n(x|\mu\delta^2 u, \delta^2 u)f_{IG}(u|\alpha, \beta)du
\]

\[
f_{VG}(x) = \int_{0}^{\infty} n(x|\theta t, \sigma^2 s)f_{GIG}(s|t, 0, \nu)ds
\]

Student-t distribution

Student-t: mixture of normal and inverse gamma

\[
f_{t}(x|f) = \frac{\Gamma\left(\frac{1}{2}(1+f)\right)}{\sqrt{\pi f} \Gamma\left(\frac{1}{2}f\right)} \left(1 + \frac{x^2}{f}\right)^{-\left(1+f\right)/2}
\]

\[
= \int_{0}^{\infty} n(x|0, u)f_{GIG}(u| -\frac{1}{2}f, f)du
\]
The NIG Lévy process

NIG Lévy density:

\[ k_{NIG}(x) = \sqrt{\frac{2}{\pi}} \delta \alpha e^{\beta x} \frac{K_1(\alpha |x|)}{|x|} \]

NIG Characteristic function:

\[ \phi_{NIG}(z|\alpha, \beta, t\delta) = \exp\left( -t\delta \left( \sqrt{\alpha^2 - (\beta + iz)^2} - \sqrt{\alpha^2 - \beta^2} \right) \right) \]

No Gaussian component: is pure jump (plus drift)

Has finite moments:

No problems with big jumps

Set \( \rho = \beta / \alpha \), then

\[ \kappa_1 = \mu + \frac{3\delta \rho}{\alpha(1-\rho^2)^{3/2}}, \]

\[ \kappa_2 = \frac{3\delta}{\alpha(1-\rho^2)^{3/2}}, \]

\[ \kappa_3 = \frac{3\delta(1+4\rho^2)}{\alpha^2(1-\rho^2)^{5/2}}, \]

\[ \kappa_3 = \frac{3(1+4\rho^2)}{\alpha^3(1-\rho^2)^{7/2}}, \]

\[ \text{skew} = \frac{3\rho}{\sqrt{\alpha \delta (1-\rho^2)^{1/4}}} \]

\[ \kappa_3 = \frac{3\rho}{\alpha \delta (1-\rho^2)^{1/2}}. \]
Time changed Brownian motion

$X_t$ a 1-dimensional semimartingale:
Representable as a time-changed Brownian motion.

$X_t = z_{A(t)}, \quad z_t$ a Brownian motion
$A(t)$ a time change.

$z_t$ is subordinated to $A(t)$

NIG: Brownian motion subordinated to IG.
VG: Brownian motion subordinated to $\Gamma$. 
Relationship to Mean-Variance Mixtures

Mean-Variance mixture:
- location $\mu$, correlation matrix $\Delta$, drift $\beta$,
- mixing distribution $F$.
- $\mu, \beta \in \mathbb{R}^d$, $\Delta$ symmetric positive definite, $|\Delta| = 1$.

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Then mean-variance mixture $M(\mu, \beta, \Delta; F)$ has
$M_{X|u} \sim N(\mu + u\beta, u\Delta)$.

Characteristic function of $M$:
$\hat{M}(z) = e^{iz'\mu} \hat{F}(z'\beta + \frac{1}{2}z'\Delta z)$

Convolutions:
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eg, Generalised hyperbolic distribution:
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$\gamma^2 = \alpha^2 - \beta' \Delta \beta$,
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Mixtures of Distributions

Write \( n(x|\mu,\Delta) \) for the normal density function. Then (\( d = 1 \))

\[
f_{GH}(x|\lambda,\alpha,\beta,\delta,\mu) = \int_0^\infty n(x|\mu+\beta u,u)f_{GIG}(u|\lambda,\delta,\sqrt{\alpha^2-\beta^2})du
\]

\[
f_{H}(x|\alpha,\beta,\delta,\mu) = \int_0^\infty n(x|\mu+\beta u,u)f_{IG}(u|\delta,\sqrt{\alpha^2-\beta^2})du
\]

\[
f_{NIG}(x|\alpha,\beta,\delta,\mu) = \int_0^\infty n(x|\mu\delta^2 u,\delta^2 u)f_{IG}(u|\alpha,\beta)du
\]

\[
f_{VG}(x) = \int_0^\infty n(x|\theta t,\sigma^2 s)f_{GIG}(s|t,0,\nu)ds
\]

Student-t distribution

Student-t: mixture of normal and inverse gamma

\[
f_t(x|f) = \frac{\Gamma\left(\frac{1}{2}(1+f)\right)}{\sqrt{\pi f\Gamma\left(\frac{1}{2}f\right)}}\left(1 + \frac{x^2}{f}\right)^{-\frac{1+f}{2}}
\]

\[
= \int_0^\infty n(x|0,u)f_{GIG}(u|^{1/2f}, f)du
\]
The Variance-Gamma Process: (Madan etc)

\( \Gamma_t^\nu \) a \( \Gamma \)-distribution, mean rate \( t \), variance rate \( \nu t \).
Its density is
\[
f^{\Gamma}(x | t, \nu) = \frac{x^{(t/\nu)-1}e^{-x/\nu}}{\nu^{t/\nu} \Gamma(t/\nu)},
\]
Laplace transform
\[
E[ \exp(-\lambda \Gamma_t^\nu) ] = (1 + \lambda \nu)^{-t/\nu}.
\]
Let \( z_t \) be a Wiener process. Define:
\[
X^{\text{VG}}(t|\sigma, \nu, \mu) = \mu \Gamma_t^\nu + \sigma z_{\Gamma_t^\nu}.
\]
Subordinated Brownian motion.

The characteristic function of \( X^{\text{VG}}(t) \) is:
\[
E[ \exp(-izX^{\text{VG}}_t) ] = (1 - i\mu \nu z + \frac{1}{2}\sigma^2 \nu z^2)^{-t/\nu}.
\]
Lévy density is:
\[
k^{\text{VG}}_t(x) = \frac{\exp(\mu x / \sigma^2)}{\nu |x|} \exp\left(-\frac{1}{\sigma} \sqrt{\frac{\mu^2}{\sigma^2} + \frac{2}{\nu} |x|}\right).
\]

The density function of \( X^{\text{VG}}_t \) is
\[
f^{\text{VG}}_t(x | \sigma, \nu, \mu) = \frac{2\exp(\mu x / \sigma^2)}{\nu^{t/\nu} \sqrt{2\pi \sigma \Gamma(\frac{t}{\nu})}} \left(\frac{x^2}{2\sigma^2 / \nu + \mu^2}\right)^{\frac{t}{2\nu} - \frac{1}{4}} K_{\frac{1}{2\nu} - \frac{1}{2}} \left(\frac{1}{\sigma} \sqrt{x^2 \left(2\sigma^2 / \nu + \mu^2\right)}\right).
\]
Modelling Interest Rates with Lévy Processes

Interest Rate Models and Lévy Processes

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Interest Rate Modelling

Use framework of Eberlein and Raible.

Denote bond prices by $P(t,T)$, accumulator account numeraire, $p_t = \exp(\int_{(0,t]} r_s ds)$. 

Let $L_t$ be a Lévy process, $\nu$ its Lévy measure, such that $\int_{|x| > 1} e^{\lambda x} \nu(dx) < \infty$, $\forall \lambda < M$, some $M$.

E & R: under the equivalent martingale measure

$$P(t,T) = P(0,T)p_t \frac{\exp\left(\int_0^T \sigma(s,T) dL_s\right)}{E\left[\exp\left(\int_0^T \sigma(s,T) dL_s\right)\right]}$$

where $0 \leq t \leq T \leq T^*$, a maximum time horizon, $\sigma > 0$, deterministic, $\sigma(T,T) = 0$, twice-differentiable, bounded. (Need to impose mild restrictions on $L_t$)

The process for $L_t$ is specified under risk-neutrality.
The process for $P(t,T)$

Laplace exponent, $e^{\kappa(u)} = E[e^{uL_1}]$,  
$$\kappa(u) = bu + \frac{1}{2}cu^2 + \int_{\mathbb{R}} (e^{ux} - 1 - ux)\nu(dx).$$

Fact: $E[\exp(-\int_{(0,t]} f(s)dL_s)] = \exp(-\int_{(0,t]} \kappa(f(s))ds)$.  
(mild conditions on $f$).

Then  
$$E[\exp(-\int_{(0,t]} \sigma_{s,T}dL_s)] = \exp(-\int_{(0,t]} \kappa(\sigma_{s,T})ds).$$

Find that  
$$P(t,T) = P(0,T)\exp(\int_{(0,t]} (r_s - \kappa(\sigma_{s,T}))ds + \int_{(0,t]} \sigma_{s,T}dL_s).$$  
$\kappa$ gives the compensator for $L_t$.

The bond process is  
$$\frac{dP(t,T)}{P(t-,T)} = r_t dt + (\frac{1}{2}c\sigma^2(t,T) - \kappa(\sigma(t,T))dt + \sigma(t,T)dL_t$$  
$$+ e^{\sigma(t,T)\Delta L_t} - 1 - \sigma(t,T)\Delta L_t.$$

If $L_t = W_t$, a Wiener process, then $\kappa(u) = \frac{1}{2}u^2$, $c = 1$.  
Reduces to usual HJM case.
The extended generalised Vasicek process

E & R show that if

- \( r_t \) is Markov
- \( \sigma \) is time homogenous (stationary)

then

\[
\sigma(t,T) \equiv \sigma(T-t) = \hat{\sigma}. \frac{1-e^{-a(T-t)}}{a}, \quad a > 0
\]

\[
= \hat{\sigma}.(T - t), \quad a = 0,
\]

Vasicek volatility is case \( a > 0 \).

Given initial term structure, forward rates \( f(0,T) \), then

\[
dr_t = a(\theta(t) - r_t)dt + \hat{\sigma}dL_t,
\]

where

\[
\rho(t) = \frac{1}{a}f_2(0,t) - \kappa'(\sigma(T-t)) \hat{\sigma}. \frac{e^{-at}}{a} + \kappa(\sigma(T-t)) + f(0,t).
\]

E & R investigate the case where \( L_t \) is hyperbolic.
NIG case

Kuan and Webber investigate pricing when $L_t$ is NIG.

Construct a lattice method,

calibrating to the initial term structure.

$L_t$ represented as time-changed Brownian motion,

$L_t = \beta h_t + z_{ht},$

where

$h_t \sim IG[\delta t, \sqrt{\alpha^2 - \beta^2}]$

is inverse Gaussian with density $f_{IG}(x | \delta, \gamma), \ x > 0,$

$f_{IG}(x | \delta, \gamma) = \frac{\delta}{\sqrt{2\pi}} x^{-3/2} \exp(\delta \gamma - \frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)).$

where $\gamma = \sqrt{\alpha^2 - \beta^2}.$

The short rate process is

$dr_t = a(\theta(t) - r_t)dt + \sigma dL_t,$

where $a$ and $\sigma$ are constants and $\theta(t)$ is deterministic.

Can choose $\theta(t)$ to fit initial term structure.
Numerical methods of option valuation.

Can use Monte Carlo for European options.

Evolve $L_t$,

$$\Delta L_t = \beta h_{\Delta t} + z h_{\Delta t},$$

where $h_{\Delta t} \sim \text{IG}[\delta \Delta t, \sqrt{\alpha^2 - \beta^2}]$,

and use Euler discretisation to evolve $r_t$,

$$\Delta r_t = a(\theta(t) - r_t)\Delta t + \sigma \Delta L_t.$$

Problems

i) Hard to use Monte Carlo for American options

ii) Can’t easily match initial term structure

iii) Hard to ensure arbitrage-free pricing,

(eg that put-call parity is satisfied).

Alternative: use a lattice method.

Can do American options easily.

Can easily match to an initial term structure.

Get arbitrage-free pricing.
Lattice Methods

Option value: expected discounted payoff (accumulator numeraire),
\[ V_0 = E_0[ \exp( - \int_0^T r_s ds ).V_T ]. \]
At time T, \( V_T = H_T \) is just the option payoff. 
\( r_t \) and \( H_T \): functions of the state variables.

Approximate continuous time state variables \( \lambda_t \)
by discrete time and space state variables, \( \lambda_{it} \), 
time step \( \Delta t \), space step \( \Delta \lambda \).
Require convergence to continuous time as \( \Delta t \to 0 \).

\( \lambda_t \) takes values \( \lambda_{ij} \) at time \( t_i = i\Delta t \), level \( \lambda_j = j\Delta \lambda \).
Require
\[ \lambda_{ij+1} - \lambda_{ij} = \Delta \lambda, \text{ for all } i,j \]
\[ \lambda_{i+1,j} = \lambda_{ij} \equiv \lambda_j, \text{ for all } i,j \]
\[ \lambda_{t_0} = \lambda_{0,0} = 0. \]
Branching probabilities:

Process is stationary with independent increments. Set
\[ p_k = \Pr[ \lambda_{t_{i+1}} = \lambda_{i+1,j+k} \mid \lambda_{t_i} = \lambda_{i,j} ], \quad -K < k < K. \]
Choose \( p_k \) so that \( \lambda_t \) approximates \( \lambda_t \).

Obtaining prices on a lattice:
- Calculate payoff at terminal time.
- Iterate back step by step.

Continuous time:
\[ V_t = E_t[ V_{t+\Delta t} \cdot \exp( -\int_t^{t+\Delta t} r_s \, ds ) ]. \]

On the lattice: (eg trinomial case, \( K = 1 \)).
For \( T = t_N \), set
\[ c_{N,j} = H_T(j\Delta \lambda) \]
\[ c_{i-1,j} = e^{-r\Delta t}(p_1c_{i,j+1} + p_0c_{i,j} + p_1c_{i,j-1}) \]
Option value is \( c_{0,0} \).
Problems

Evolving forward: (obtaining probabilities).
Base on true moments, or good approximations.
(Avoid using Euler discretisation).

Discounting back. If $r_t$ not constant:
Must use Brownian Bridge discounting.

Sample low probability regions?
Prune to get rid of wasted branches.

Non-uniform convergence?
Use terminal correction.
A Trinomial Lattice

State variable evolved on a trinomial lattice. Discretise calendar time as $t_i = t_0 + i\Delta t$, $i = 0,\ldots,N$.

Nodes on the lattice are labelled $(i,j)$, where $i$ labels time, $j$ labels level, $j = -N,\ldots,N$.

Branching from node $(i,j)$ is to

$$\{ (i+1, j-1), (i+1, j), (i+1, j+1) \}$$

with probabilities $p^d$, $p^m$, $p^u$. 
At a typical node

Single state variable $r_t$.
Trinomial lattice: $r_t$ branches up or across or down.

Ordinary idea:
Choose probabilities $p_u$, $p_m$ and $p_d$ so that
the discrete variable has
the same expected value and
the same variance
as the continuous variable.
Doesn’t work for Lévy processes.
Problem

Wiener process: defined by its 1st and 2nd moments. Lévy process: isn’t.

Define a lattice directly?
   Requires very high order, very wide branching (Kellezi and Webber)

Alternative: (Kuan and Webber)
   Combine Monte Carlo with lattices.

Generate random lattices.
   On each lattice, compute option value
   Take the average from many random lattices.
Modelling Interest Rates with Lévy Processes

The Random Lattice

Grace Kuan and Nick Webber
The Random Lattice

Evolve a lattice up to time calendar $T$ in $N$ time steps. Calendar time step is $\Delta t = T/N$.

Set up a trinomial lattice using random step size. Time-changed time step is $\Delta \tau_i$ with

$$\tau_{i+1} - \tau_i = \Delta \tau_i \sim IG[\, \delta \Delta t, \, \sqrt{\alpha^2 - \beta^2} \, ]$$

Key point: Conditional on $\Delta \tau_i$,

$$L_{\Delta \tau_i} \text{ is normal, } L_{\Delta \tau_i} \sim N[\, -\beta \Delta \tau_i, \, \Delta \tau_i \, ]$$

Use a modified Hull and White lattice, with variable time step.

Two stage method:

First: build a tree for the process

$$dr_t = -ar_t dt + \sigma dL_t$$

Second: Incorporate the drift, $\theta(t)$, by offsetting the nodes of the tree.
The Stage One Tree

Use time step $\Delta t = T/N$, base moments on $\Delta \tau_i$. set $\Delta r = \sigma \sqrt{\kappa \Delta \tau}$, where $\Delta \tau = \max \{\Delta \tau_i\}$, $\kappa > 1$, (so that probabilities remain positive).

Construct a tree with:

- **time steps:**
  
  $0$, $\Delta t$, $2\Delta t$, $3\Delta t$, ..., $N\Delta t = T$.

- **space steps:**
  
  $-j_m \Delta r, \ldots, -2\Delta r, -\Delta r, 0, \Delta r, 2\Delta r, \ldots, j_m \Delta r$.

The node for time $i\Delta t$, rate $j\Delta r$, is labelled $(i,j)$. 
Probabilities

At each node have three branches.
Assign a probability to each branch: \( p^u, p^m, p^d \).

At each node fix the probabilities so that the discrete density on the tree matches the continuous density of \( r \).

Conditional mean is:
\[
m_{i,j} = \mathbb{E}[\Delta r \mid (i,j)] = -a(j\Delta r)\Delta t + \sigma \beta \Delta \tau_i,
\]
Conditional variance is:
\[
v_{i,j} = \text{var}[\Delta r \mid (i,j)] = \sigma^2 \Delta \tau_i,
\]

Solve to get:
\[
p^u = \frac{1}{2\Delta r^2}(m_{i,j}^2 + v_{i,j} + m_{i,j} \Delta r),
\]
\[
p^m = \frac{1}{\Delta r^2}(m_{i,j}^2 + v_{i,j}),
\]
\[
p^d = \frac{1}{2\Delta r^2}(m_{i,j}^2 + v_{i,j} - m_{i,j} \Delta r).
\]
(Have altered branching when \(|j| = j_{\text{max}}\).
In practice lattice truncated before hitting level \( j_{\text{max}} \).)
Second Stage: Incorporate the Drift

Given initial term structure to match to,

\[ P_i = \text{value at time } 0 \text{ of pdb maturing at time } i \Delta t. \]
\[ R_i = \text{the spot rate for time } i \Delta t, \quad P_i = e^{-i \Delta t R_i}. \]

Construct lattice for \( r \) by offsetting lattice for \( r \).

ie, at time \( i \Delta t \) displace the nodes at \( j \Delta r \) to \( \alpha_i + j \Delta r \).

Use continuous time \( \alpha \)? Doesn’t work on lattice.

Write \( Q_{i,j} \) for value at time 0 of pure security for state \((i,j)\). Pays out

1 at time \( i \Delta t \) if \( r \) has value \( r_{i,j} = \alpha_i + j \Delta r \),
0 otherwise.
Forward induction.

Iterate forward finding successive $\alpha_i$.
Have two recurrence relationships.

Replication of pdb value:

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \cdot \exp(- (\alpha_m + j\Delta r) \Delta t)$$

where $j = -n_m, \ldots, n_m$ for the nodes at time $m\Delta t$.

Iterated expectation:

$$Q_{m+1,j} = \sum_k Q_{m,k} \cdot q_{k,j} \cdot \exp(- (\alpha_m + k\Delta r) \Delta t)$$

where $q_{k,j}$ is the probability of going from node (m,k) to node (m+1,j).

Given $\alpha_{m-1}$ and $Q_{m,j}$, obtain $\alpha_m$ and $Q_{m+1,j}$.
But $Q_{0,0} = 1$ and $\alpha_0 = R_1$. Now finish the tree.
Valuation

On the lattice, if know payoff $V_{N,j}$ at time $T = t_N$, then option value $V$ at time 0 is

$$V = \sum_{j=-N}^{N} Q_{N,j} V_{N,j}.$$ 

Using the random lattice.

Generate $M$ random lattices,

get $M$ option values $V_k$, $k = 1, \ldots, M$.

Set option value to be $V = \frac{1}{M} \sum_{k=1}^{M} V_k$. 
Results
Compute caplet prices on the lattice, benchmark using a Monte Carlo method.

Caplet: Maturity time, $T = 1$, Libor has tenor 0.25. Payoff is

$$H_T = \max(0, L_T - X)$$

where $L_T$ is Libor at time $T$, $X$ is exercise rate.

Monte Carlo Method:
Benchmark to the term structure.
Recovers well (to 4dp) with $N = 100$, $M = 25000$.

For caplet:
To get underlying libor,
simulate further 20 paths from time 1 to time 1.25.
Use Michael, Schucany and Hass (76) algorithm to generate IG variates.
Caplet Prices on the Lattice
Random lattice with N = 40 time steps to 1.25 years.

Parameter values: (a little arbitrary...)
NIG: $\alpha = 10$, $\beta = 2$, $\delta = 10$,
Short rate: $a = 0.01$, $\sigma = 0.02$, $r_0 = 0.06$.
Initial term structure flat at 0.06.

<table>
<thead>
<tr>
<th>Strike, X</th>
<th>Monte Carlo, M = 25,000</th>
<th>Monte Carlo, M = 4,000</th>
<th>Monte Carlo, M = 5,000</th>
<th>Monte Carlo, M = 6,000</th>
<th>Monte Carlo, M = 7,000</th>
<th>Monte Carlo, M = 8,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.005258 (7.2E-06)</td>
<td>0.005224 (9.3E-07)</td>
<td>0.005225 (8.7E-07)</td>
<td>0.005225 (7.8E-07)</td>
<td>0.005224 (7.2E-07)</td>
<td>0.005224 (6.8E-07)</td>
</tr>
<tr>
<td>0.06</td>
<td>0.001999 (9.9E-06)</td>
<td>0.001971 (1.6E-06)</td>
<td>0.001976 (1.4E-06)</td>
<td>0.001976 (1.3E-06)</td>
<td>0.001976 (1.2E-06)</td>
<td>0.001976 (1.0E-06)</td>
</tr>
<tr>
<td>0.08</td>
<td>0.000441 (5.5E-06)</td>
<td>0.000440 (1.1E-06)</td>
<td>0.000139 (8.9E-07)</td>
<td>0.000438 (8.2E-07)</td>
<td>0.000440 (7.8E-07)</td>
<td>0.000440 (7.3E-07)</td>
</tr>
</tbody>
</table>

Black’s implied volatility

| Strike, X | 0.4768 | 0.3526 | 0.3022 |

Run times: Random lattice: 257 secs (M = 8000)
Monte Carlo: 247 secs (M = 25000)
Monte Carlo used 100 time steps over 1.25 years.
Both implemented in C on a Unix Ultra Enterprise 450
Caplet Prices on the Lattice

Parameter values: (a little arbitrary...)

NIG: \( \alpha = 36, \ \beta = 2, \ \delta = 10, \)
Short rate: \( a = 0.2, \ \sigma = 0.02, \ r_0 = 0.06. \)
Initial term structure flat at 0.06.

<table>
<thead>
<tr>
<th>Strike, X</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lattice, M = 4000</td>
<td>0.004812 (1.9E-05)</td>
<td>0.002613 (1.7E-05)</td>
<td>0.000944 (1.0E-05)</td>
<td>0.000184 (3.4E-06)</td>
<td>0.000017 (6.2E-07)</td>
</tr>
<tr>
<td>Lattice, M = 5000</td>
<td>0.004807 (1.6E-05)</td>
<td>0.002613 (1.5E-05)</td>
<td>0.000945 (9.4E-06)</td>
<td>0.000185 (3.3E-06)</td>
<td>0.000017 (5.4E-07)</td>
</tr>
<tr>
<td>Lattice, M = 6000</td>
<td>0.004810 (1.5E-05)</td>
<td>0.002611 (1.4E-05)</td>
<td>0.000944 (8.9E-06)</td>
<td>0.00184 (3.3E-06)</td>
<td>0.000017 (5.2E-07)</td>
</tr>
<tr>
<td>Lattice, M = 7000</td>
<td>0.004810 (1.4E-05)</td>
<td>0.002612 (1.3E-05)</td>
<td>0.000944 (7.9E-06)</td>
<td>0.000184 (2.8E-06)</td>
<td>0.000017 (4.7E-07)</td>
</tr>
<tr>
<td>Lattice, M = 8000</td>
<td>0.004814 (1.3E-05)</td>
<td>0.002613 (1.2E-05)</td>
<td>0.000945 (7.5E-06)</td>
<td>0.000184 (2.7E-06)</td>
<td>0.000017 (4.5E-07)</td>
</tr>
<tr>
<td>Blacks vol, (M = 8000)</td>
<td>0.3197</td>
<td>0.2131</td>
<td>0.1704</td>
<td>0.1517</td>
<td>0.1407</td>
</tr>
</tbody>
</table>

Get pronounced volatility skew.
Caplet Prices on the Lattice

Parameter values: (a little arbitrary...)

\[ \alpha = 36, \quad \beta = 2, \quad \delta = 10, \]

Short rate: \[ a = 0.2, \quad \sigma = 0.02, \quad r_0 = 0.06. \]

Initial term structure flat at 0.06.

Prices for various times to maturity:

<table>
<thead>
<tr>
<th>Strike, X</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
</tr>
</thead>
<tbody>
<tr>
<td>T = 0.25</td>
<td>0.005039</td>
<td>0.002586</td>
<td>0.000554</td>
<td>0.000015</td>
<td>5.38E-08</td>
</tr>
<tr>
<td></td>
<td>(1.26E-06)</td>
<td>(5.32E-07)</td>
<td>(3.81E-07)</td>
<td>(3.24E-07)</td>
<td>(6.78E-07)</td>
</tr>
<tr>
<td>T = 0.5</td>
<td>0.004963</td>
<td>0.002594</td>
<td>0.000747</td>
<td>0.000068</td>
<td>1.20E-06</td>
</tr>
<tr>
<td></td>
<td>(3.24E-06)</td>
<td>(1.25E-05)</td>
<td>(9.37E-06)</td>
<td>(1.52E-07)</td>
<td>(1.06E-08)</td>
</tr>
<tr>
<td>T = 0.75</td>
<td>0.004872</td>
<td>0.002603</td>
<td>0.000864</td>
<td>0.000132</td>
<td>7.20E-06</td>
</tr>
<tr>
<td></td>
<td>(1.93E-05)</td>
<td>(1.86E-05)</td>
<td>(1.13E-05)</td>
<td>(3.13E-06)</td>
<td>(4.89E-07)</td>
</tr>
<tr>
<td>T = 1</td>
<td>0.004810</td>
<td>0.002612</td>
<td>0.000944</td>
<td>0.000184</td>
<td>0.00017</td>
</tr>
<tr>
<td></td>
<td>(1.72E-05)</td>
<td>(1.60E-05)</td>
<td>(1.11E-05)</td>
<td>(3.37E-06)</td>
<td>(6.02E-07)</td>
</tr>
</tbody>
</table>

Blacks implied vols. for various times to maturity:

<table>
<thead>
<tr>
<th>Blacks Implied Volatilities</th>
<th>Strike, X</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
</tr>
</thead>
<tbody>
<tr>
<td>T = 0.25</td>
<td>0.6417</td>
<td>0.3377</td>
<td>0.1909</td>
<td>0.1640</td>
<td>0.1601</td>
<td></td>
</tr>
<tr>
<td>T = 0.5</td>
<td>0.4537</td>
<td>0.2604</td>
<td>0.1848</td>
<td>0.1590</td>
<td>0.1484</td>
<td></td>
</tr>
<tr>
<td>T = 0.75</td>
<td>0.3606</td>
<td>0.2386</td>
<td>0.1772</td>
<td>0.15670</td>
<td>0.1438</td>
<td></td>
</tr>
<tr>
<td>T = 1</td>
<td>0.3177</td>
<td>0.2128</td>
<td>0.1793</td>
<td>0.1517</td>
<td>0.1407</td>
<td></td>
</tr>
</tbody>
</table>
Plot of Blacks Implied Volatilities
Effect of blipping parameter values

Blacks implied volatility

Strike

Base case — Blip alpha
- Blip beta — Blip delta
Effect of changes in Parameter Values
Observations

Random lattice:
More accurate than Monte Carlo.
Similar run times at greatest accuracy.

Run times for random lattice:

<table>
<thead>
<tr>
<th>M</th>
<th>run time (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
<td>85</td>
</tr>
<tr>
<td>5000</td>
<td>124</td>
</tr>
<tr>
<td>6000</td>
<td>158</td>
</tr>
<tr>
<td>7000</td>
<td>197</td>
</tr>
<tr>
<td>8000</td>
<td>257</td>
</tr>
</tbody>
</table>

Can get strong volatility skews.

Effect of blipping parameter values

\[ \alpha \rightarrow 40, \quad \text{Tilt down, short end remains stable} \]
\[ \beta \rightarrow 2.5, \quad \text{Tilt down, long end remains stable} \]
\[ \delta \rightarrow 12, \quad \text{Shift upwards} \]
Conclusions

Demonstrated that
   i) NIG processes feasible in E & R context
   ii) Can implement on a lattice

Can calibrate to
   An initial term structure.
   Potentially, to Blacks implied volatilities.

Lattice method: in principle
   Implementable for any process expressible as
   time-changed Brownian motion.

No reason not to explore use of NIG processes
and other Lévy processes in interest rate modelling.